

XXIV. *Some Properties of the Sum of the Divisors of Numbers.*

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1. **L**ET the equation $\overline{x-1} \cdot \overline{x^2-1} \cdot \overline{x^3-1} \cdot \overline{x^4-1} \cdot \overline{x^5-1} \cdot \dots \overline{x^n-1} = x^b - px^{b-1} + qx^{b-2} - rx^{b-3} + sx^{b-4} - \&c. = x^b - x^{b-1} - x^{b-2} + x^{b-5} + x^{b-7} - x^{b-12} - x^{b-15} + x^{b-22} + x^{b-26} - x^{b-35} - x^{b-40} + x^{b-51} + x^{b-57} - \&c. \dots x^{b-n} \pm \&c. = A = 0$. The signs + and - proceed alternately by pairs unto the term x^{b-n} . The co-efficients of all the terms to the above mentioned (x^{b-n}) will be +1, -1 or 0; they will be +1, when multiplied into x^{b-v} , where $v = \frac{3z^2+z}{2}$ or $= \frac{3z^2-z}{2}$, and z an even number; but -1, if z be an uneven number; in all other cases they will be = 0.

The numbers 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, &c. subtracted from b may be collected from the addition of the numbers 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, &c. which consist of two arithmetical serieses 1, 3, 5, 7, 9, 11, &c. 1, 2, 3, 4, 5, 6, 7, &c. intermixed.

2. The sum of any power (m) of each of the roots in the equation $A=0$ will be $S(m)$, where $S(m)$ denotes the sum of all the divisors of the number m , if m be not greater than n .

Cor. Hence (by the rule for finding the sum of (m) powers of each of the roots from the sum of the inferior powers and co-efficients of the given equation) may be deduced $S(m) =$

$pS(m-1) - qS(m-2) + rS(m-3) - sS(m-4) + tS(m-5) - \&c. = S(m-1) + S(m-2) - S(m-5) - S(m-7) + S(m-12) + S(m-15) - S(m-22) - S(m-26) + \&c.$ which is the property of the sum of divisors invented by the late M. EULER.

Cor. By substituting for $S(m-1)$, $S(m-2)$, &c. their values $S(m-2) + S(m-3) - S(m-6) - S(m-8) + \&c.$, $S(m-3) + S(m-4) - S(m-7) - S(m-9) + \&c.$ &c. in the given equation $S(m) = S(m-1) + S(m-2) - S(m-5) - S(m-7) + \&c.$ may be acquired an expression for the sum $S(m)$ in terms of the sums of the divisors of numbers less than $m-1$, $m-2$, &c. — the same method may be used for a similar purpose in some of the following propositions.

Cor. By the rule for finding the sum of the contents of every (m) roots from the sums of the powers of each of the roots may be deduced the equation $\pm 1 \cdot 2 \cdot 3 \cdot 4 \dots m$,

$$\begin{aligned} \text{or } 0 = 1 - m \cdot \frac{m-1}{2} S(2) + m \cdot \overline{m-1} \cdot \frac{m-2}{3} S(3) \\ - m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \frac{m-3}{4} S(4) + \&c. \\ + m \cdot \overline{m-1} \cdot \frac{m-2}{2} \cdot \frac{m-3}{2^2} S((2))^2 - \&c. \end{aligned}$$

in which the sum of the divisors of any number m is expressed by the sums of the divisors of the inferior numbers $m-1$, $m-2$, &c. and their powers. If v be an even number, then $\pm 1 \cdot 2 \cdot 3 \dots m$ will have the same sign as the coefficient; if uneven, the contrary; but if the coefficient $= 0$, then will the content $1 \cdot 2 \cdot 3 \dots m$ vanish. The law of this series is given in the *Meditationes Algebraicæ*.

3. Let H be the number of different ways by which the sum of any two numbers $1, 2, 3, 4, \dots m-2, m-1$, can become $= m$; H' the number of ways by which the sum of any three of the above-mentioned numbers can make m ; H'' , H''' , H'''' , &c. the number of ways by which the sum of any four, five, six,

&c. of the above-mentioned numbers is $= m$ respectively; then will $1 - H + H' - H'' + H''' - \&c. = \pm 1$ or 0 . Let $m = \frac{3z^2 \pm z}{2}$, and it will be $+1$ or -1 , according as z is an odd or even number, in all other cases it will be $= 0$.

P A R T II.

1. Let the equation be $\overline{x-1} \cdot \overline{x^2-1} \cdot \overline{x^3-1} \cdot \overline{x^5-1} \cdot \overline{x^7-1} \cdot \overline{x^{11}-1} \cdot \overline{x^{13}-1} \cdot \overline{x^{17}-1} \dots \overline{x^n-1} \cdot \&c. = x^{b'} - px^{b'-1} + qx^{b'-2} - rx^{b'-3} + sx^{b'-4} - \&c. = x^{b'} - x^{b'-1} - x^{b'-2} + x^{b'-4} + x^{b'-8} - x^{b'-10} - x^{b'-11} + x^{b'-12} + x^{b'-16} - x^{b'-17} - x^{b'-19} + x^{b'-20} - x^{b'-23} + 2x^{b'-24} - x^{b'-26} - x^{b'-27} + x^{b'-28}, \&c. = A' = 0$; the sum of any power (m) of each of the roots in the equation $A' = 0$ will be $S'(m)$, where $S'(m)$ denotes the sum of all the prime divisors of the number m , and m is not greater than n .

Cor. Hence, by the rule before-mentioned $S'(m) = S'(m-1) + S'(m-2) - S'(m-4) - S'(m-8) + S'(m-10) + S'(m-11) - S'(m-12) - S'(m-16) + S'(m-17) + S'(m-19) - S'(m-20) + S'(m-23) - 2S'(m-24) + S'(m-26) + S'(m-27) - S'(m-28) + S'(m-29), \&c.$

If in this, or the preceding, or subsequent analogous cases $S(m-r)$, or $S'(m-r)$, or $S^l(m-r)$, becomes $S(0)$, or $S'(0)$, or $S^l(0)$; for $S(0)$, or $S'(0)$, or $S^l(0)$, always substitute r .

Cor. Let L be the co-efficient of the term $x^{b'-m}$; then, by the above-mentioned series contained in the Meditationes Algebraicæ, will $1 \cdot 2 \cdot 3 \cdot 4 \dots m \times L = 1 - m \cdot \frac{m-1}{2} S'(2) + m \cdot \overline{m-1} \cdot \frac{m-2}{3} \times S'(3) - m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \frac{m-3}{4} \times S'(4) + \&c.$

$+ m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \frac{m-3}{8} \times S'((2))^2$

$[- \&c.]$

- &c. be an equation, which expresses a relation between the prime divisors of the numbers 1, 2, 3, 4 . . . $m-1$, m , and their powers.

Cor. The co-efficient L = the difference between the two respective numbers of different ways that m can be formed adding the prime numbers 1, 2, 3, 5, 7, 11, 13, 19, &c. the one with, and the other without, 2.

P A R T III.

1. Let an equation $\overline{x^a - 1} \cdot \overline{x^b - 1} \cdot \overline{x^c - 1} \cdot \overline{x^d - 1} \times \&c. = x^b - px^{b-1} + qx^{b-2} - rx^{b-3} + \&c. = 0$; then will the sum of the (m) powers of each of its roots be the sum of all the divisors of m , that can be found amongst the numbers α , β , γ , δ , &c.

2. The co-efficient of the term x^{b-m} will be the difference between the two respective numbers of different ways, that the number (m) can be formed from the addition of the numbers α , β , γ , δ , &c.; the one containing in it an odd number of the even numbers contained in α , β , γ , δ , &c.; the other not.

P A R T IV.

1. Let $\overline{x^l - 1} \cdot \overline{x^{2l} - 1} \cdot \overline{x^{3l} - 1} \cdot \overline{x^{4l} - 1} \cdot \dots \cdot \overline{x^{nl} - 1} \cdot \&c. = x^b - px^{b-l} + qx^{b-2l} - rx^{b-3l} + \&c. = x^b - x^{b-l} - x^{b-2l} + x^{b-3l} + x^{b-4l} - x^{b-5l} - x^{b-6l} + \&c. = B = 0$, of which equation all the co-efficients are the same as in case the first, and consequently ± 1 or 0 to the term (x^{b-nl}) .

2. The sum of any power $l \times m$ of each of the roots of the equation $B = 0$ will be $S'(m)$, where $S'(m)$ denotes the sum of the divisors of m , which are divisible by l .

Cor.

Cor. Hence $S^l(m) = S^l(m-1) + S^l(m-2l) - S^l(m-5l) - S^l(m-7l) + S^l(m-12l) + S^l(m-15l) - S^l(m-22l) - S^l(m-26l) + \&c.$; the law of the series has been given in Case 1.

Cor. The sum of all the divisors of m not divisible by $l = S(m) - S^l(m) = S(m-1) - S^l(m-1) + (S(m-2) - S^l(m-2l)) - (S(m-5) - S^l(m-5l)) - (S(m-7) - S^l(m-7l)) + \&c.$

A similar rule may be predicated of the sum of the divisors not divisible by the numbers $a, b, c, d, \&c.$; for the sum of the divisors of the number (m) divisible by $a, b, c, d, e, \&c.$, where $a, b, c, d, e, \&c.$ are prime to each other $= (S^a(m) + S^b(m) + S^c(m) + S^d(m) + S^e(m) + \&c.) - ((S^{a \times b}(m) + S^{a \times c}(m) + S^{b \times c}(m) + S^{a \times d}(m) + S^{b \times d}(m) + S^{c \times d}(m) + S^{a \times e}(m) + \&c.) + (S^{a \times b \times c}(m) + S^{a \times b \times d}(m) + S^{a \times c \times d}(m) + S^{b \times c \times d}(m) + S^{a+b}(m) + \&c.)) - ((S^{a+c+d}(m) + S^{a+b+c+e}(m) + \&c.)) + (S^{a+b+c+d+e}(m) + \&c.)) - \&c. = l =$ the sum of all the divisors of $m \dots$ divisible by $a, b, c, d, e, \&c.$ respectively added together, $-$ the sum of all the divisors of m divisible by the products $(ab, ac, bc, \&c.)$ of any two of the quantities $a, b, c, d, \&c.$ $+$ the sum of all the divisors of m divisible by the contents $(abc, abd, acd, bcd, \&c.)$ of every three of the quantities $a, b, c, d, \&c.$ $-$ the sum of all the divisors of m divisible by the contents of every four of the abovementioned quantities $a, b, c, d, \&c.$ $+$ and so on, and consequently $S(m) - C$ is the sum required.

The principles given in the former parts may be applied to this, and extended to equations of which the factors have the formula $x^a \pm k$; and from the sum of the inferior powers of each of the roots, and the co-efficients may be collected the sum of the superior; the same may be performed by the co-efficients only, $\&c.$

P A R T V.

1. $S(\alpha \times \beta) = \alpha \times S(\beta) +$ sum of all the divisors of β not divisible by $\alpha = \beta \times S(\alpha) +$ sum of all the divisors of α not divisible by β .

2. $S'(\alpha \times \beta) = \alpha \times S'(\beta) +$ sum of all the divisors of β divisible by l but not by $\alpha = \&c.$

3. $S(\alpha \times \beta \times \gamma \times \delta \times \&c.) = \alpha \times S(\beta \times \gamma \times \delta \times \epsilon, \&c.) +$ sum of all the divisors of $\beta \times \gamma \times \delta \times \epsilon, \&c.$ not divisible by $\alpha = \alpha \times \beta \times S(\gamma \times \delta \times \epsilon, \&c.) +$ sum of all the divisors of $\beta \times \gamma \times \delta \times \epsilon, \&c.$ not divisible by $\alpha + \alpha \times$ sum of all the divisors of $\gamma \times \delta \times \epsilon, \&c.$ not divisible by $\beta = \alpha \times \beta \times \gamma \times S(\delta \times \epsilon, \&c.) +$ sum of all the divisors of $\beta \times \gamma \times \delta \times \epsilon, \&c.$ not divisible by $\alpha + \alpha \times$ sum of all the divisors of $\gamma \times \delta \times \epsilon, \&c.$ not divisible by $\beta + \alpha \times \beta \times$ sum of all the divisors of $\delta \times \epsilon, \&c.$ not divisible by $\gamma = \alpha \times \beta \times \gamma \times \delta \times S(\epsilon, \&c.) +$ sum of all the divisors of $\beta \times \gamma \times \delta \times \epsilon, \&c.$ not divisible by $\alpha + \alpha \times$ sum of all the divisors of $\gamma \delta \epsilon, \&c.$ not divisible by $\beta + \alpha \times \beta \times$ sum of all the divisors of $\delta \epsilon, \&c.$ not divisible by $\gamma + \alpha \times \beta \times \gamma \times$ sum of all the divisors of $\epsilon, \&c.$ not divisible by $\delta = \&c.$ The law of the series is manifest. The letters $\alpha, \beta, \gamma, \delta, \&c.$ which are not contained between the parentheses, denote prime numbers.

Cor. If some of the letters $\alpha, \beta, \gamma, \delta, \&c.$ be substituted for others, and others for them, the equations resulting will be just, and consequently many new equations may be deduced.

If in the preceding equations for S be wrote S' , and for the sum of all the divisors of a certain quantity not divisible by a prime number (α , or β , or $\gamma, \&c.$) be wrote the sum of all the

the divisors of that quantity not divisible by the same prime number, but divisible by l ; the propositions resulting will be true.

These equations may be applied to the equations given in the preceding parts, and from thence many others deduced.

